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Absolutely continuous functions in $\mathbb{R}^n$

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Abstract

For each $0 < \alpha < 1$ we consider a natural n-dimensional extension of the classical notion of absolute continuous function. We compare it with the Malý's and Hencl's definitions. It follows that each α-absolute continuous function is continuous, weak differentiable with gradient in L^n, differentiable almost everywhere and satisfies the formula on change of variables.

Keywords: α-regular intervals; Absolutely continuous functions; Sobolev spaces

1. Introduction

The well-known notion of absolutely continuous functions of one variable has been generalized to more variables in many ways motivated by different problems (see [5] for the definitions due to S. Banach and L. Tonelli).

The problem of finding regular functions in the Sobolev space $W^{1,n}$ induced, recently, J. Malý [4] and S. Hencl [2], to introduce the following definition.

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Definition 1. Let $0 < \lambda \leqslant 1$. A mapping f from an open subset Ω of $\mathbb{R}^n$ to $\mathbb{R}^l$ is said to be *n, λ-absolutely continuous* if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sum_i \operatorname{osc}^n\bigl(f, B(\mathbf{x}_i, \lambda r_i)\bigr) < \varepsilon,$$

for each nonoverlapping finite family $\{B(\mathbf{x}_i, r_i)\}$ of closed balls in Ω with

$$\sum_i \mathcal{L}^n\bigl(B(\mathbf{x}_i, r_i)\bigr) < \delta.$$

Remark that, for $\lambda \in (0, 1)$, it is possible to change, in Definition 1, from balls to cubes without affecting the resulting classes of functions (denoted by $\mathcal{B}\text{-}AC^n_\lambda(\Omega, \mathbb{R}^l)$ and $\mathcal{Q}\text{-}AC^n_\lambda(\Omega, \mathbb{R}^l)$, respectively); but $\mathcal{B}\text{-}AC^n_1(\Omega, \mathbb{R}^l)$ and $\mathcal{Q}\text{-}AC^n_1(\Omega, \mathbb{R}^l)$ are different (see [1] and [3]).

Remark, moreover, that balls and cubes coincide, on the line, with the intervals, and that the classical notion of absolutely continuous functions makes use of the increments, instead of the oscillations. It is then natural to consider, even for functions of more variables, an analog, more familiar, definition of absolute continuity:

Definition 2. Let $0 < \alpha < 1$. A mapping f from an open subset Ω of $\mathbb{R}^n$ to $\mathbb{R}^l$ is said to be *α-absolutely continuous* (briefly $f \in \alpha\text{-}AC^{(n)}(\Omega, \mathbb{R}^l)$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_i \bigl| f(\mathbf{b}_i) - f(\mathbf{a}_i)\bigr|^n < \varepsilon,$$

for each disjoint finite family of α-regular intervals $\{[\mathbf{a}_i, \mathbf{b}_i] \subset \Omega\}$ with

$$\sum_i \mathcal{L}^n\bigl([\mathbf{a}_i, \mathbf{b}_i]\bigr) < \delta.$$

The aim of this paper is to prove that

$$\mathcal{Q}\text{-}AC^n_1\bigl(\Omega, \mathbb{R}^l\bigr) \subsetneqq \alpha\text{-}AC^{(n)}\bigl(\Omega, \mathbb{R}^l\bigr) \subsetneqq \mathcal{B}\text{-}AC^n_\lambda\bigl(\Omega, \mathbb{R}^l\bigr),$$

for each $0 < \alpha, \lambda < 1$.

Consequently, by [2, Theorems 3.4 and 3.5] the following theorems hold.

Theorem 3. *If $f\colon \Omega \to \mathbb{R}^l$ is α-absolutely continuous, with $0 < \alpha < 1$, then f belongs to the Sobolev space $W^{1,n}(\Omega, \mathbb{R}^l)$. Moreover, f is continuous on Ω and differentiable almost everywhere.*

Theorem 4. *If $f\colon \Omega \to \mathbb{R}^l$, $m \geqslant n$ is α-absolutely continuous, with $0 < \alpha < 1$, if E is a measurable subset of Ω, and if $u : E \to \mathbb{R}^l$ is measurable with $u|J_f| \in L^1(E)$, then*

$$\int_E u(x)\bigl|J_f(x)\bigr|\,dx = \int_{f(E)} \sum_{\{x \in E:\ f(x)=y\}} u(x)\,d\mathcal{H}^n(y).$$

2. Preliminaries

The elements of $\mathbb{R}^n$ are denoted by bold face lower case letters. The coordinates of $\mathbf{a} \in \mathbb{R}^n$ are denoted by $a_1, \dots, a_n$, the euclidean norm of $\mathbf{a}$ is denoted by $|\mathbf{a}|$, the closed ball with center $\mathbf{a}$ and radius r is denoted by $B(\mathbf{a}, r)$.

Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $a_h < b_h$, $h = 1, \dots, n$, we write $\mathbf{a} < \mathbf{b}$ and we denote by $[\mathbf{a}, \mathbf{b}]$ the (n-dimensional) interval having $\mathbf{a}$ and $\mathbf{b}$ as extreme points; i.e.,

$$[\mathbf{a}, \mathbf{b}] = \left\{\mathbf{t} \in \mathbb{R}^n\colon\ a_h \leqslant t_h \leqslant b_h,\ h = 1, \dots, n\right\}.$$

For $0 < \alpha < 1$ we say that $[\mathbf{a}, \mathbf{b}]$ is α-regular whenever

$$r[\mathbf{a}, \mathbf{b}] = \frac{\mathcal{L}^n([\mathbf{a}, \mathbf{b}])}{(\max_h |a_h - b_h|)^n} \geqslant \alpha.$$

We set

$$\tau_\alpha = \frac{4\sqrt{n}}{1 - \sqrt[n]{\alpha}}.$$

Lemma 5. *Let $\mathbf{b} \in B(\mathbf{a}, r)$, with $\mathbf{b} \neq \mathbf{a}$. Then there exists $\mathbf{c} \in B(\mathbf{a}, \tau_\alpha r)$ such that*

(i) $\mathbf{c} < \mathbf{a}$, $r[\mathbf{c}, \mathbf{a}] > \alpha$, $[\mathbf{c}, \mathbf{a}] \subset B(\mathbf{a}, \tau_\alpha r)$;
(ii) $\mathbf{c} < \mathbf{b}$, $r[\mathbf{c}, \mathbf{b}] > \alpha$, $[\mathbf{c}, \mathbf{b}] \subset B(\mathbf{a}, \tau_\alpha r)$.

Proof. Define

$$\mathbf{c} = -\frac{(1 + \sqrt[n]{\alpha})r}{1 - \sqrt[n]{\alpha}}\mathbf{1} + \mathbf{a},$$

where $\mathbf{1} = (1, \dots, 1)$. Then

$$|\mathbf{c} - \mathbf{a}| = \frac{(1 + \sqrt[n]{\alpha})r}{1 - \sqrt[n]{\alpha}}\sqrt{n};$$

hence $\mathbf{c} \in B(\mathbf{a}, \tau_\alpha r)$.

(i) By definition of $\mathbf{c}$ it follows

$$c_h = -\frac{(1 + \sqrt[n]{\alpha})r}{1 - \sqrt[n]{\alpha}} + a_h < -r + a_h < a_h, \tag{1}$$

for $h = 1, \dots, n$, hence $\mathbf{c} < \mathbf{a}$. Moreover, since

$$a_h - c_h = \frac{(1 + \sqrt[n]{\alpha})r}{1 - \sqrt[n]{\alpha}}, \quad h = 1, \dots, n,$$

we have $r[\mathbf{c}, \mathbf{a}] = 1 > \alpha$. The inclusion $[\mathbf{c}, \mathbf{a}] \subset B(\mathbf{a}, \tau_\alpha r)$ is immediate.

(ii) Since $\mathbf{b} \in B(\mathbf{a}, r)$ we have $a_h - r \leqslant b_h \leqslant a_h + r$, $h = 1, \dots, n$. Therefore (1) implies

$$c_h < -r + a_h \leqslant b_h, \quad h = 1, \dots, n.$$

Hence $\mathbf{c} < \mathbf{b}$. Moreover, we have

$$b_h - c_h = b_h - a_h + \frac{1 + \sqrt[n]{\alpha}}{1 - \sqrt[n]{\alpha}}r \leqslant r + \frac{1 + \sqrt[n]{\alpha}}{1 - \sqrt[n]{\alpha}}r = \frac{2r}{1 - \sqrt[n]{\alpha}},$$

and

$$b_h - c_h \geqslant -r + \frac{1+\sqrt[n]{\alpha}}{1-\sqrt[n]{\alpha}} r = \frac{2r}{1-\sqrt[n]{\alpha}} \sqrt[n]{\alpha}.$$

Therefore

$$r[\mathbf{c}, \mathbf{b}] \geqslant \left(\frac{2r}{1-\sqrt[n]{\alpha}} \sqrt[n]{\alpha} \right)^n \bigg/ \left(\frac{2r}{1-\sqrt[n]{\alpha}} \right)^n = \alpha.$$

Now, for each $\mathbf{u} \in [\mathbf{c}, \mathbf{b}]$, we have

$$\begin{aligned} |\mathbf{u} - \mathbf{a}| &\leqslant |\mathbf{u} - \mathbf{b}| + |\mathbf{b} - \mathbf{a}| \leqslant |\mathbf{c} - \mathbf{b}| + r \leqslant |\mathbf{c} - \mathbf{a}| + |\mathbf{a} - \mathbf{b}| + r \\ &\leqslant \left(\frac{1+\sqrt[n]{\alpha}}{1-\sqrt[n]{\alpha}} \sqrt{n} + 2 \right) r < \frac{4\sqrt{n}}{1-\sqrt[n]{\alpha}} r = \tau_\alpha r; \end{aligned}$$

thus $[\mathbf{c}, \mathbf{b}] \subset B(\mathbf{a}, \tau_\alpha r)$. $\square$

Lemma 6. *Let $0 < \alpha < 1$. If $[\mathbf{a}, \mathbf{b}]$ is an α-regular interval, then there exist nonoverlapping closed cubes $Q_1, \dots, Q_k \subset [\mathbf{a}, \mathbf{b}]$, with $k \leqslant 1/\alpha + 1$, and points $\mathbf{x}_0, \dots, \mathbf{x}_k$ such that $\mathbf{x}_0 = \mathbf{a}$, $\mathbf{x}_k = \mathbf{b}$ and $\mathbf{x}_{i-1}, \mathbf{x}_i \in Q_i$, for $i = 1, 2, \dots, k$.*

Proof. Let $[\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and let

$$m = \min_h \{b_h - a_h\}, \qquad M = \max_h \{b_h - a_h\}.$$

Since $[\mathbf{a}, \mathbf{b}]$ is α-regular, we have

$$\frac{m}{M} = \frac{M^{n-1} \cdot m}{M^n} \geqslant \frac{\mathcal{L}^n([\mathbf{a}, \mathbf{b}])}{M^n} \geqslant \alpha.$$

Hence

$$\frac{M}{m} \leqslant \frac{1}{\alpha}. \tag{2}$$

Now, for $h = 1, \dots, n$, let s_h be the integer part of $|b_h - a_h|/m$, and let $\mathbf{s} = (s_1, s_2, \dots, s_n)$. Then, the interval

$$I = [\mathbf{a}, \mathbf{a} + m\mathbf{s}] = [a_1, a_1 + ms_1] \times \cdots \times [a_n, a_n + ms_n]$$

is a subset of $[\mathbf{a}, \mathbf{b}]$. Moreover, I is union of $s_1 s_2 \cdots s_n$ nonoverlapping cubes of side m; namely

$$I = \bigcup_{j_1=0}^{s_1-1} \cdots \bigcup_{j_n=0}^{s_n-1} \big[a_1 + j_1 m, a_1 + (j_1+1)m\big] \times \cdots \times \big[a_n + j_n m, a_n + (j_n+1)m\big].$$

Let $k = \max_h\{s_h\} + 1$, $\tilde{M} = \max_h\{|b_h - (a_h + s_h m)|\}$, and define

$$Q_k = [b_1 - \tilde{M}, b_1] \times \cdots \times [b_n - \tilde{M}, b_n].$$

Clearly the cube Q_k is disjoint with the interior of I, and $\mathbf{a} + m\mathbf{s}$, $\mathbf{b} \in Q_k$. Now, by the definition of s_h it follows that $s_h \leqslant (b_h - a_h)/m$, $h = 1, 2, \dots, n$. Therefore, by (2), we have

$$\max_h\{s_h\} \leqslant \frac{1}{m} \max_h\{b_h - a_h\} = \frac{M}{m} \leqslant \frac{1}{\alpha}.$$

Hence

$$k \leqslant 1/\alpha + 1.$$

Now, assume for simplicity that $1 = s_1 < s_2 < \cdots < s_n = k - 1$ and take

$$\begin{aligned}
Q_1 &= [a_1, a_1 + m] \times [a_2, a_2 + m] \times \cdots \times [a_n, a_n + m], \\
Q_2 &= [a_1, a_1 + m] \times [a_2 + m, a_2 + 2m] \times [a_3 + m, a_3 + 2m] \\
&\quad \times \cdots \times [a_n + m, a_n + 2m], \\
&\qquad \cdots\cdots\cdots \\
Q_{s_2} &= [a_1, a_1 + m] \times \big[a_2 + (s_2 - 1)m, a_2 + s_2 m\big] \\
&\quad \times \big[a_3 + (s_2 - 1)m, a_3 + s_2 m\big] \times \cdots \times \big[a_n + (s_2 - 1)m, a_n + s_2 m\big], \\
Q_{s_2+1} &= [a_1, a_1 + m] \times \big[a_2 + (s_2 - 1)m, a_2 + s_2 m\big] \\
&\quad \times \big[a_3 + s_2 m, a_3 + (s_2 + 1)m\big] \times \cdots \times \big[a_n + s_2 m, a_n + (s_2 + 1)m\big], \\
&\qquad \cdots\cdots\cdots \\
Q_{s_{n-1}} &= [a_1, a_1 + m] \times \big[a_2 + (s_2 - 1)m, a_2 + s_2 m\big] \\
&\quad \times \cdots \times \big[a_{n-1} + (s_{n-1} - 1)m, a_{n-1} + s_{n-1} m\big] \\
&\quad \times \big[a_n + (s_{n-1} - 1)m, a_n + s_{n-1} m\big], \\
&\qquad \cdots\cdots\cdots \\
Q_{s_n} &= [a_1, a_1 + m] \times \big[a_2 + (s_2 - 1)m, a_2 + s_2 m\big] \\
&\quad \times \cdots \times \big[a_n + (s_n - 1)m, a_n + s_n m\big],
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_0 &= (a_1, a_2, \ldots, a_n) = \mathbf{a}, \\
\mathbf{x}_1 &= (a_1 + m, a_2 + m, \ldots, a_n + m), \\
\mathbf{x}_2 &= (a_1 + m, a_2 + 2m, a_3 + m, \ldots, a_n + m), \\
&\qquad \cdots\cdots\cdots \\
\mathbf{x}_{s_2} &= (a_1 + m, a_2 + s_2 m, a_3 + 2m, \ldots, a_n + 2m), \\
&\qquad \cdots\cdots\cdots \\
\mathbf{x}_{s_2+1} &= \big(a_1 + m, a_2 + s_2 m, a_3 + (s_2 + 1)m, \ldots, a_n + (s_2 + 1)m\big), \\
&\qquad \cdots\cdots\cdots \\
\mathbf{x}_{s_n} = \mathbf{x}_{k-1} &= (a_1 + m, a_2 + s_2 m, a_3 + s_3 m \ldots, a_n + s_n m) = \mathbf{a} + m\mathbf{s}, \\
\mathbf{x}_k &= (b_1, b_2, \ldots, b_n) = \mathbf{b}.
\end{aligned}$$

It is easy to check that $Q_1, \ldots, Q_k$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k$ satisfy all the conditions required by the statement. Hence the proof is complete. □

3. The embedding theorems

Theorem 7. *Let $0 < \alpha < 1$ and let $0 < \lambda < 1$. If $f \in \alpha\text{-}AC^{(n)}(\Omega, \mathbb{R}^l)$, then $f \in AC^n_\lambda(\Omega, \mathbb{R}^l)$.*

Proof. Given $\varepsilon > 0$, let $\delta > 0$ be such that

$$\sum_i \bigl| f(\mathbf{b}_i) - f(\mathbf{a}_i)\bigr|^n < \frac{1}{2 \cdot 4^n}\varepsilon,$$

for each nonoverlapping finite family of α-regular intervals $\{[\mathbf{a}_i, \mathbf{b}_i] \subset \Omega\}$ with

$$\sum_i \mathcal{L}^n\bigl([\mathbf{a}_i, \mathbf{b}_i]\bigr) < \delta.$$

Now let $\{B(\mathbf{x}_i, r_i)\}$ be a finite number of nonoverlapping balls in Ω such that

$$\sum_i \mathcal{L}^n\bigl(B(\mathbf{x}_i, r_i)\bigr) < \frac{\mathcal{L}^n(B(\mathbf{0}, 1))\delta}{2^n}.$$

For each i, we take $\mathbf{a}_i, \mathbf{b}_i \in B(\mathbf{x}_i, r_i/(2\tau_\alpha + 1))$ such that

$$\operatorname{osc}\bigl(f, B\bigl(\mathbf{x}_i, r_i/(2\tau_\alpha + 1)\bigr)\bigr) \leqslant 2\bigl| f(\mathbf{b}_i) - f(\mathbf{a}_i)\bigr|.$$

Since $\mathbf{b}_i \in B(\mathbf{a}_i, 2r_i/(2\tau_\alpha + 1))$, by Lemma 5 there exists

$$\mathbf{c}_i \in B\bigl(\mathbf{a}_i, 2r_i\tau_\alpha/(2\tau_\alpha + 1)\bigr)$$

such that the intervals $[\mathbf{c}_i, \mathbf{a}_i]$ and $[\mathbf{c}_i, \mathbf{b}_i]$ are α-regular and

$$[\mathbf{c}_i, \mathbf{a}_i] \subset B\bigl(\mathbf{a}_i, 2r_i\tau_\alpha/(2\tau_\alpha + 1)\bigr), \qquad [\mathbf{c}_i, \mathbf{b}_i] \subset B\bigl(\mathbf{a}_i, 2r_i\tau_\alpha/(2\tau_\alpha + 1)\bigr).$$

Moreover, it is easy to check that

$$B\bigl(\mathbf{a}_i, 2r_i\tau_\alpha/(2\tau_\alpha + 1)\bigr) \subset B(\mathbf{x}_i, r_i).$$

Consequently, the intervals $\{[\mathbf{c}_i, \mathbf{a}_i]\}$ are nonoverlapping, and

$$\sum_i \mathcal{L}^n\bigl([\mathbf{c}_i, \mathbf{a}_i]\bigr) \leqslant \sum_i |\mathbf{a}_i - \mathbf{c}_i|^n \leqslant 2^n \sum_i r_i^n = 2^n \frac{\sum_i \mathcal{L}^n(B(\mathbf{x}_i, r_i))}{\mathcal{L}^n(B(\mathbf{0}, 1))} < \delta.$$

Thus

$$\sum_i \bigl| f(\mathbf{a}_i) - f(\mathbf{c}_i)\bigr|^n < \frac{1}{2 \cdot 4^n}\varepsilon.$$

Analogously we prove that

$$\sum_i \bigl| f(\mathbf{b}_i) - f(\mathbf{c}_i)\bigr|^n < \frac{1}{2 \cdot 4^n}\varepsilon.$$

Therefore,

$$
\begin{aligned}
&\sum_i \operatorname{osc}^n\big(f, B\big(\mathbf{x}_i, r_i/(2\tau_\alpha+1)\big)\big)\\
&\quad\leqslant 2^n \sum_i \big|f(\mathbf{b}_i)-f(\mathbf{a}_i)\big|^n\\
&\quad\leqslant 4^n\left(\sum_i \big|f(\mathbf{a}_i)-f(\mathbf{c}_i)\big|^n+\sum_i \big|f(\mathbf{b}_i)-f(\mathbf{c}_i)\big|^n\right)<\varepsilon.
\end{aligned}
$$

Hence $f \in AC^n_{1/(2\tau_\alpha+1)}(\Omega,\mathbb{R}^l)$. To end the proof it is enough to apply [2, Theorem 3.2]. $\square$

Theorem 8. *If $f \in \mathcal{Q}\text{-}AC^n_1(\Omega,\mathbb{R}^l)$, then $f \in \alpha\text{-}AC^{(n)}(\Omega,\mathbb{R}^l)$, for each $0<\alpha<1$.*

Proof. Given $\varepsilon>0$ let $\delta>0$ be such that

$$
\sum_i \operatorname{osc}^n(f,Q_i)<\frac{\varepsilon}{(1/\alpha+1)^n}
$$

for each nonoverlapping system of closed cubes $\{Q_i\}_i \subset \Omega$ with

$$
\sum_i \mathcal{L}^n(Q_i)<\delta.
$$

Given $0<\alpha<1$, let $\mathcal{F}\equiv\{[\mathbf{a}_i,\mathbf{b}_i]\}_i$ be a finite family of α-regular nonoverlapping intervals in Ω such that

$$
\sum_i \mathcal{L}^n\big([\mathbf{a}_i,\mathbf{b}_i]\big)<\delta.
$$

For a fixed i, by Lemma 6 there exist nonoverlapping closed cubes $Q_1^{(i)},\dots,Q_{k(i)}^{(i)} \subset [\mathbf{a}_i,\mathbf{b}_i]$, with $k(i)<1/\alpha+1$, and points $\mathbf{x}_0^{(i)},\dots,\mathbf{x}_{k(i)}^{(i)}$ such that $\mathbf{x}_0^{(i)}=\mathbf{a}_i$, $\mathbf{x}_{k(i)}^{(i)}=\mathbf{b}_i$ and $\mathbf{x}_{(h-1)}^{(i)},\mathbf{x}_h^{(i)}\in Q_h^{(i)}$, for $h=1,2\dots,k(i)$. Then, since

$$
\sum_i\sum_{h=1}^{k(i)} \mathcal{L}^n\big(Q_h^{(i)}\big)\leqslant \sum_i \mathcal{L}^n\big([\mathbf{a}_i,\mathbf{b}_i]\big)<\delta,
$$

we have

$$
\begin{aligned}
\sum_i \big|f(\mathbf{b}_i)-f(\mathbf{a}_i)\big|^n &\leqslant \sum_i\sum_{h=1}^{k(i)}\big(k(i)\big)^n\big|f\big(\mathbf{x}_h^{(i)}\big)-f\big(\mathbf{x}_{h-1}^{(i)}\big)\big|^n\\
&\leqslant \sum_i\sum_{h=1}^{k(i)}(1/\alpha+1)^n \operatorname{osc}^n\big(f,Q_h^{(i)}\big)\\
&\leqslant (1/\alpha+1)^n\frac{\varepsilon}{(1/\alpha+1)^n}=\varepsilon.
\end{aligned}
$$

This completes the proof. $\square$

4. Examples

(1) We start by proving that α-$AC^{(n)}(\Omega, \mathbb{R}^l)$ is strictly contained into $AC^n_\lambda(\Omega, \mathbb{R}^l)$, for each $0 < \alpha, \lambda < 1$. For simplicity, we assume $n = 2$, $\Omega = B(0, 1/8)$, and $l = 1$.

Let us consider the function

$$f(\mathbf{x}) = \begin{cases} \dfrac{\sin(\log(\log \frac{1}{|\mathbf{x}|}))}{\sqrt{\log(\log \frac{1}{|\mathbf{x}|})}} & \text{if } \mathbf{x} \neq 0, \mathbf{x} \in B(0, 1/8), \\ 0 & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

S. Hencl [2, Theorem 6.4] proved that $f \in AC^2_\lambda(B(0, 1/8), \mathbb{R})$, for each $0 < \lambda < 1$. Now, for $k = 1, 2, \ldots,$ we define

$$\mathbf{a}_k = \frac{1}{\sqrt{2}}\left(e^{-e^{\pi/2+2k\pi}}, e^{-e^{\pi/2+2k\pi}}\right), \qquad \mathbf{b}_k = \frac{1}{\sqrt{2}}\left(e^{-e^{2k\pi}}, e^{-e^{2k\pi}}\right).$$

It is easy to see that $\mathbf{a}_k < \mathbf{b}_k < \mathbf{a}_{k-1}$, for each k. Then the squares $[\mathbf{a}_k, \mathbf{b}_k]$, $k = 1, 2, \ldots,$ are pairwise disjoint. Moreover, since

$$\sum_k \left| f(\mathbf{b}_k) - f(\mathbf{a}_k)\right|^2 = \sum_k \left| \frac{1}{\sqrt{\pi/2 + 2k\pi}} - 0\right|^2 = \infty,$$

we can conclude that $f \notin \alpha$-$AC(B(0, 1/8), \mathbb{R})$, for each $0 < \alpha < 1$.

(2) Now we prove that $\mathcal{Q}$-$AC^n_1(\Omega, \mathbb{R}^l)$ is strictly contained into α-$AC^{(n)}(\Omega, \mathbb{R}^l)$, for each $0 < \alpha < 1$. For simplicity, we assume $n = 2$, $l = 1$, and $1/2 \leqslant \alpha < 1$. The idea of the construction goes back to [1].

We define f as the sum of a totally convergent series of non-negative continuous functions f_m such that the support of each f_m is covered by the union of

$$r_m = 4^{m-1}(m-1)!\,m!$$

pairwise disjoint squares

$$Q_{m1}, Q_{m2}, \ldots, Q_{mr_m},$$

and such that

$$\max f_m = \omega_m = \frac{1}{2^m m!}.$$

The square Q_{11} is arbitrarily chosen in Ω. Assumed that, for a given m, the functions $f_1, f_2, \ldots, f_{m-1}$ and the squares Q_{hk}, $1 \leqslant h \leqslant m$, $1 \leqslant k \leqslant r_h$, have been defined, let us define f_m and the squares $Q_{(m+1)j}$, $j = 1, \ldots, r_{m+1}$ as follows: for a fixed $k \in \{1, \ldots, r_m\}$ we put an horizontal and a vertical line through the midpoint $\mathbf{o} = (o_1, o_2)$ of the square Q_{mk}; then, denoted by $2d$ the side of Q_{mk}, for every interval

$$[a_i, b_i] = \begin{cases} [d/2^i, d/2^{i-1}] & \text{if } i = 1, 2, \ldots, m-1, \\ [0, d/2^{m-1}] & \text{if } i = m, \end{cases}$$

we put $4(m + 1)$ small squares $Q_{(m+1)j}$ inside the strip

$$M_i = \left\{(x, y) \in \mathbb{R}^2\colon\ p(x, y) \in \left[\frac{2a_i + b_i}{3}, \frac{2b_i + a_i}{3}\right]\right\},$$

where $p(x, y) = |x - o_1| + |y - o_2|$. More precisely, we put $2(m + 1)$ small squares $Q_{(m+1)j}$ inside the strip $M_i \cap \{(x, y)\colon x > o_1,\ y > o_2\}$, and the remaining $2(m + 1)$ small squares $Q_{(m+1)j}$ inside the strip $M_i \cap \{(x, y)\colon x \leqslant o_1,\ y \leqslant o_2\}$. We also require that the distribution of the squares inside the strips is the following:

(1) if m is even, then we put
- $m/2 + 1$ squares $Q_{(m+1)j}$ inside each of the following strips:

$$\begin{cases} M_i \cap \big\{(x, y)\colon x > o_1,\ 0 < y - o_2 < d/2^{i+2}\big\}; \\ M_i \cap \big\{(x, y)\colon x < o_1,\ 0 < o_2 - y < d/2^{i+2}\big\}; \\ M_i \cap \big\{(x, y)\colon 0 < x - o_1 < d/2^{i+2},\ y > o_2\big\}; \\ M_i \cap \big\{(x, y)\colon 0 < o_1 - x < d/2^{i+2},\ y < o_2\big\}; \end{cases} \tag{3}$$

- $m/2$ squares $Q_{(m+1)j}$ into the interval $[\mathbf{p}_1, \mathbf{p}_2]$ with extreme points $\mathbf{p}_1$, $\mathbf{p}_2$ lying on the line $y = o_2 + (x - o_1)/2$ and satisfying the conditions $p(\mathbf{p}_1) = \frac{4}{3} \cdot \frac{d}{2^i}$, $p(\mathbf{p}_2) = \frac{5}{3} \cdot \frac{d}{2^i}$;
- $m/2$ squares $Q_{(m+1)j}$ into the interval $[\mathbf{q}_1, \mathbf{q}_2]$ with extreme points $\mathbf{q}_1$, $\mathbf{q}_2$ lying on the line $y = o_2 + 2(x - o_1)$ and satisfying the conditions $p(\mathbf{q}_1) = \frac{4}{3} \cdot \frac{d}{2^i}$ and $p(\mathbf{q}_2) = \frac{5}{3} \cdot \frac{d}{2^i}$;

(2) if m is odd, then we put:
- $(m + 1)/2$ squares $Q_{(m+1)j}$ inside each of the strips (3);
- $(m + 1)/2$ squares $Q_{(m+1)j}$ into the interval $[\mathbf{p}_1, \mathbf{p}_2]$ with extreme points $\mathbf{p}_1$, $\mathbf{p}_2$ lying on the line $y = o_2 + (x - o_1)/2$ and satisfying the conditions $p(\mathbf{p}_1) = \frac{4}{3} \cdot \frac{d}{2^i}$ and $p(\mathbf{p}_2) = \frac{5}{3} \cdot \frac{d}{2^i}$;
- $(m + 1)/2$ squares $Q_{(m+1)j}$ into the interval $[\mathbf{q}_1, \mathbf{q}_2]$ with extreme points $\mathbf{q}_1$, $\mathbf{q}_2$ lying on the line $y = o_2 + 2(x - o_1)$ and satisfying the conditions $p(\mathbf{q}_1) = \frac{4}{3} \cdot \frac{d}{2^i}$ and $p(\mathbf{q}_2) = \frac{5}{3} \cdot \frac{d}{2^i}$.

On Q_{mk} the function f_m is defined such that $f_m(\mathbf{x}) = \tilde{f}_m(p(\mathbf{x}))$, where

$$\tilde{f}_m(p) = \begin{cases} 0 & \text{if } p \geqslant d; \\ i\omega_m/m & \text{if } p = d/2^i,\ i = 1, 2, \ldots, m - 1; \\ \omega_m & \text{if } p = 0. \end{cases}$$

Moreover, on the intervals $[a_i, b_i]$, $i = 1, 2, \ldots, m$, we define $\tilde{f}_m$ by

$$\tilde{f}_m(p) = \begin{cases} (\tilde{f}_m(a_i) + \tilde{f}_m(b_i))/2 & \text{if } p \in [(2a_i + b_i)/3, (2b_i + a_i)/3]; \\ \text{linear} & \text{on } [a_i, (2a_i + b_i)/3] \text{ and } [(2b_i + a_i)/3, b_i]. \end{cases}$$

The function $f = \sum_m f_m$ is not $\mathcal{Q}$-$AC_1^2(\Omega, \mathbb{R})$. In fact, for each $m \in \mathbb{N}$ and $k = 1, 2, \ldots, r_m$, let T_{mk} be the right-lower quarter of the square Q_{mk}. Then the system $\{T_{mk}\}$ form a pairwise disjoint system of squares, and $\operatorname{osc}(f, T_{mk}) = \omega_m$, for each $m \in \mathbb{N}$ and $k = 1, 2, \ldots, r_m$. Therefore

$$\sum_m \sum_k \operatorname{osc}^2(f, T_{mk}) = \sum_m \sum_k \omega_m^2 = \sum_m r_m \omega_m^2 = \sum_m \frac{1}{4m} = \infty.$$

Thus $f \notin \mathcal{Q}\text{-}AC_1^2(\Omega,\mathbb{R})$.

Now we prove that $f \in \alpha\text{-}AC^2(\Omega,\mathbb{R})$. By an adaptation of [1, Lemma 3], we can prove that for every α-regular interval $I = [\mathbf{a},\mathbf{b}]$ there exists an index $m = m(I)$, for which

$$\left|f(\mathbf{b}) - f(\mathbf{a})\right|^2 \leqslant 16\left|f_{m(I)}(\mathbf{b}) - f_{m(I)}(\mathbf{a})\right|^2. \tag{4}$$

Let

$$\mathcal{D}_1 = \left\{I = [\mathbf{a},\mathbf{b}]\colon\ I \text{ is } \alpha\text{-regular and } \left|f_{m(I)}(\mathbf{b}) - f_{m(I)}(\mathbf{a})\right| \leqslant 9\frac{\omega_{m(I)}}{m(I)}\right\},$$

$$\mathcal{D}_2 = \left\{I = [\mathbf{a},\mathbf{b}]\colon\ I \text{ is } \alpha\text{-regular and } \left|f_{m(I)}(\mathbf{b}) - f_{m(I)}(\mathbf{a})\right| \geqslant 9\frac{\omega_{m(I)}}{m(I)}\right\}.$$

We will prove that there exist two measures μ_1 and μ_2, absolutely continuous with respect to the Lebesgue measure, such that

$$\left|f_{m(I)}(\mathbf{b}) - f_{m(I)}(\mathbf{a})\right|^2 \leqslant \mu_1\big([\mathbf{a},\mathbf{b}]\big), \tag{5}$$

for each $[\mathbf{a},\mathbf{b}] \in \mathcal{D}_1$, and

$$\left|f_{m(I)}(\mathbf{b}) - f_{m(I)}(\mathbf{a})\right|^2 \leqslant \mu_2\big([\mathbf{a},\mathbf{b}]\big), \tag{6}$$

for each $[\mathbf{a},\mathbf{b}] \in \mathcal{D}_2$.

The absolute continuity of μ_1 and μ_2 implies that, given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\mu_1\left(\bigcup_j [\mathbf{a}_j,\mathbf{b}_j]\right) < \frac{\varepsilon}{32}, \qquad \mu_2\left(\bigcup_j [\mathbf{a}_j,\mathbf{b}_j]\right) < \frac{\varepsilon}{32},$$

for each nonoverlapping finite family $\{[\mathbf{a}_j,\mathbf{b}_j]\}$ of α-regular intervals such that $\mathcal{L}^n(\bigcup_j [\mathbf{a}_j,\mathbf{b}_j]) < \delta$. Hence, by (4)–(6), we obtain

$$\begin{aligned}\sum_j \left|f(\mathbf{b}_j) - f(\mathbf{a}_j)\right|^2 &\leqslant 16\sum_j \left|f_{m([\mathbf{a}_j,\mathbf{b}_j])}(\mathbf{b}_j) - f_{m([\mathbf{a}_j,\mathbf{b}_j])}(\mathbf{a}_j)\right|^2 \\ &< 16\sum_j \mu_1\big([\mathbf{a}_j,\mathbf{b}_j]\big) + 16\sum_j \mu_2\big([\mathbf{a}_j,\mathbf{b}_j]\big) \\ &= 16\mu_1\left(\bigcup_j [\mathbf{a}_j,\mathbf{b}_j]\right) + 16\mu_2\left(\bigcup_j [\mathbf{a}_j,\mathbf{b}_j]\right) < \varepsilon,\end{aligned}$$

which proves that f is $\alpha\text{-}AC^2(\Omega,\mathbb{R})$.

So, to complete the proof, we have to prove the existence of measures μ_1 and μ_2.

Existence of measure μ_1

For a fixed interval $I = [\mathbf{a},\mathbf{b}] \in \mathcal{D}_1$, let $m = m(I)$ be such that $|f(\mathbf{b}) - f(\mathbf{a})| \leqslant 4|f_{m(I)}(\mathbf{b}) - f_{m(I)}(\mathbf{a})|$, and let $I^* = [\mathbf{a}^*,\mathbf{b}^*]$ be the smallest α-regular sub-interval of I such that $|f_m(\mathbf{b}^*) - f_m(\mathbf{a}^*)| = |f_m(\mathbf{b}) - f_m(\mathbf{a})|$. Then $I^* \subset Q_{mk}$, for some k.

For $1 \leqslant i \leqslant m-9$ we set

$$S_i = \left\{\mathbf{x} \in \mathbb{R}^2\colon\ p(\mathbf{x}) \in \bigcup_{j=i}^{i+9} [a_j,b_j]\right\}.$$

Since $|f_m(\mathbf{b}^*) - f_m(\mathbf{a}^*)| \leqslant (9\omega_m)/m$, we have $\mathbf{a}^*, \mathbf{b}^* \in S_i$ for some integer $1 \leqslant i \leqslant m-9$. Now remark that $\tilde{f}_m$ is Lipschitz on $\bigcup_{j=i}^{i+9}[a_j, b_j]$ with Lipschitz constant

$$\frac{\omega_m/2m}{\frac{1}{3}\cdot\frac{\sqrt{2}}{2}\cdot(b_{i+9}-a_{i+9})} \leqslant \frac{\omega_m/2m}{\frac{1}{3}\cdot\frac{\sqrt{2}}{2}\cdot(d/2^{i+9-1}-d/2^{i+9})}$$
$$= 3\cdot 2^{9-1}\cdot\sqrt{2}\cdot\frac{\omega_m}{m}\cdot\frac{1}{d/2^i}.$$

Therefore

$$\left|f_m(\mathbf{b}^*) - f_m(\mathbf{a}^*)\right|^2 \leqslant 9\cdot 2^{17}\cdot\frac{\omega_m^2}{m^2}\cdot\frac{(\operatorname{diam} I^*)^2}{(d/2^i)^2}.$$

Hence, since

$$\left(\operatorname{diam} I^*\right)^2 \leqslant \left(1/\alpha^2+1\right)\mathcal{L}^2\left(I^*\right) \quad \text{and} \quad \mathcal{L}^2(S_i) < 8\cdot\left(d/2^i\right)^2,$$

we have

$$\left|f_m(\mathbf{b}^*) - f_m(\mathbf{a}^*)\right|^2 \leqslant 9\cdot 2^{20}\cdot\frac{\omega_m^2}{m^2}\cdot\left(1/\alpha^2+1\right)\cdot\frac{\mathcal{L}^2(I^*)}{\mathcal{L}^2(S_i)}$$
$$= \int_{I^*} 9\cdot 2^{20}\cdot\left(1/\alpha^2+1\right)\cdot\frac{\omega_m^2}{m^2}\cdot\frac{1}{\mathcal{L}^2(S_i)}.$$

Thus if we set $\mu_1 = \int g$, where

$$g(\mathbf{x}) = 9\cdot 2^{20}\cdot\left(1/\alpha^2+1\right)\cdot\sum_{m=1}^{\infty}\sum_{k=1}^{r_m}\sum_{i=1}^{m-9}\left(\frac{\omega_m^2}{m^2}\cdot\frac{\chi_{S_i}(\mathbf{x})}{\mathcal{L}^2(S_i)}\right) \in L^1\left(\mathbb{R}^2\right),$$

then μ_1 satisfies condition (5) for each interval $[\mathbf{a}, \mathbf{b}] \in \mathcal{D}_1$ (we can follow [1, p. 154] for details).

Existence of measure μ_2

Let μ_2 be an absolutely continuous measure for which

$$\mu_2(Q_{mk}) = \frac{4}{m\cdot r_m}, \quad m\in\mathbb{N},\ k = 1,\dots,r_m.$$

This measure exists because

$$\sum_{j=1}^{r_{m+1}/r_m}\mu_2(Q_{(m+1)j}) = \frac{4}{(m+1)\cdot r_m} < \frac{4}{m\cdot r_m} = \mu_2(Q_{mk}),$$

and

$$\sum_{k=1}^{r_m}\mu_2(Q_{mk}) = \frac{4}{m} \to 0.$$

As before, for a fixed interval $I = [\mathbf{a}, \mathbf{b}] \in \mathcal{D}_2$, let $m = m(I)$ be such that $|f(\mathbf{b}) - f(\mathbf{a})| \leqslant 4|f_m(\mathbf{b}) - f_m(\mathbf{a})|$ and let $I^* = [\mathbf{a}^*, \mathbf{b}^*]$ be the smallest α-regular sub-interval of I such that $|f_m(\mathbf{b}^*) - f_m(\mathbf{a}^*)| = |f_m(\mathbf{b}) - f_m(\mathbf{a})|$. Then $I^* \subset Q_{mk}$ for some k. Let

$$\beta = \frac{|f(\mathbf{b}) - f(\mathbf{a})| \cdot m}{\omega_m},$$

and let $\mathbf{o}$ be the midpoint of Q_{mk}. Since $I \in \mathcal{D}_2$, we have $\beta \geqslant 9$.

First of all let us show that I^* cannot belongs to the lower-right quarter and to the upper-left quarter of Q_{mk}. By the symmetry it is enough to prove that I^* is not in the lower-right quarter of Q_{mk}.

If $p(\mathbf{a}^*) < p(\mathbf{b}^*)$, it is trivial to observe that $b_2 - a_2 < b_1 - a_1$ and $b_1 - a_1 > p(\mathbf{b}^*) - p(\mathbf{a}^*)$, where $(a_1, a_2) = \mathbf{a}^*$ and $(b_1, b_2) = \mathbf{b}^*$. Therefore, denoted by j the smallest integer such that $p(\mathbf{b}^*) \geqslant d/2^j$, and by i the biggest integer such that $p(\mathbf{a}^*) \leqslant d/2^{j+i}$, we get

$$\alpha \leqslant \frac{b_2 - a_2}{b_1 - a_1} \leqslant \frac{d/2^{j+i}}{\frac{d}{2^j} - \frac{d}{2^{j+i}}} = \frac{1}{2^i - 1}.$$

Hence $2^i \leqslant 1/\alpha + 1$. Now, by the choice of i and j it follows $\beta < (j + i + 1) - (j - 1) = i + 2$. Thus we get the following contradiction: $\beta < i + 2 \leqslant \log_2(1/\alpha + 1) + 2 < 9$.

Analogously we can proceed in the case $p(\mathbf{b}^*) < p(\mathbf{a}^*)$.

Now let us assume that I^* intersects either the horizontal or the vertical line for the midpoint $\mathbf{o} = (o_1, o_2)$ of Q_{mk}. By the symmetry, it is enough to consider the case in which I^* intersects the horizontal line in the right half of Q_{mk}. For simplicity we can assume that $o_1 = o_2 = 0$. Then we have $a_1 > 0, a_2 < 0$ and $b_1 > 0, b_2 > 0$.

First of all let us show that we cannot find any α-regular interval with $p(\mathbf{b}^*) < p(\mathbf{a}^*)$. Indeed, let us denote by j the smallest integer such that $p(\mathbf{a}^*) \geqslant d/2^j$, and by i the biggest integer such that $p(\mathbf{b}^*) \leqslant d/2^{j+i}$. Then $i \geqslant 9$, and $b_1 - a_1 < \frac{d}{2^{j+i}}$, $b_2 - a_2 > \frac{d}{2^j} - \frac{d}{2^{j+i}}$. So by the α-regularity of I^* we get the following contradiction:

$$\frac{1}{2} < \frac{1/2^{j+i}}{1/2^j - 1/2^{j+i}} = \frac{1}{2^i - 1} \leqslant \frac{1}{2^9 - 1} < \frac{1}{2}.$$

Consequently we have $p(\mathbf{a}^*) < p(\mathbf{b}^*)$. Now, denote by j the smallest integer such that $p(\mathbf{b}^*) \geqslant d/2^j$, and by i the biggest integer such that $p(\mathbf{a}^*) \leqslant d/2^{j+i}$. Then $b_1 - a_1 \geqslant \frac{d}{2^j} - \frac{d}{2^{j+i}}$, $|a_2| < \frac{d}{2^{j+i}}$, and $i \geqslant 9$. Therefore the assumption $b_2 \leqslant d/2^{j+2}$ implies

$$\frac{b_2 - a_2}{b_1 - a_1} \leqslant \frac{\frac{d}{2^{j+2}} + \frac{d}{2^{j+i}}}{\frac{d}{2^j} - \frac{d}{2^{j+i}}} = \frac{2^{i-2} + 1}{2^i - 1} < \frac{1}{2} \leqslant \alpha,$$

which is in contradiction with the α-regularity of I^*. Thus we have $b_2 > d/2^{j+2}$. Moreover, the assumption $b_1 < d/2^{j+1}$ implies $b_1 - a_1 < d/2^{j+1}$ and $b_2 > d/2^{j+1}$. Therefore

$$b_1 - a_1 < \frac{d}{2^{j+1}} < b_2 < b_2 - a_2.$$

Thus, since I^* is α-regular, we have $b_2 - a_2 \leqslant 2(b_1 - a_1)$; hence $d/2^{j+1} < b_2 - a_2 \leqslant 2(b_1 - a_1)$. So $d/2^{j+2} < b_1 - a_1 < b_1$. In conclusion we have $b_2 > d/2^{j+2}$ and $b_1 > d/2^{j+2}$. This implies

$$I^* \supset \bigcup_{k=3}^{i} M_{j+k} \cap \big\{(x, y)\colon\ x > 0, 0 < y < d/2^{j+k+2}\big\}.$$

Consequently, the interval I^* contains at least $(i-2)\frac{m+1}{2}$ squares $Q_{(m+1)h}$, with $1 \leqslant h \leqslant r_{m+1}$. Therefore,

$$\mu_2\big([\mathbf{a}^*, \mathbf{b}^*]\big) \geqslant (i-2) \cdot \frac{m+1}{2} \cdot \frac{4}{(m+1)r_{m+1}} = \frac{2(i-2)}{m+1}\omega_m^2.$$

Thus, since $\beta \leqslant m$ and $\beta \leqslant i+2$, we have

$$\begin{aligned} \big|f(\mathbf{b}^*) - f(\mathbf{a}^*)\big|^2 = \frac{\beta^2}{m^2} \cdot \omega_m^2 &\leqslant \frac{\beta}{m} \cdot \frac{\beta}{m} \cdot \frac{m+1}{2(i-2)}\mu_2\big([\mathbf{a}^*, \mathbf{b}^*]\big) \\ &\leqslant \frac{i+2}{2(i-2)} \cdot \frac{m+1}{m}\mu_2\big([\mathbf{a}^*, \mathbf{b}^*]\big). \end{aligned}$$

Moreover, since $m \geqslant i \geqslant 9$, we have

$$mi \geqslant 9m > 7m + 2 > 6m + i + 2.$$

Then

$$\begin{aligned} (i+2)(m+1) &= mi + 2m + i + 2 = mi + (6m + i + 2) - 4m \\ &< 2mi - 4m = 2m(i-2). \end{aligned}$$

Hence $|f(\mathbf{b}^*) - f(\mathbf{a}^*)|^2 < \mu_2([\mathbf{a}^*, \mathbf{b}^*])$.

Now let us consider the case in which I^* belongs to the upper-right quarter of Q_{mk}. As before, we can assume, for simplicity, that $\mathbf{o} = (0, 0)$ and we denote by j the smallest integer such that $p(\mathbf{b}^*) \geqslant d/2^j$, and by i the biggest integer such that $p(\mathbf{a}^*) \leqslant d/2^{j+i}$.

For $1 \leqslant v \leqslant i$, we set

$$S_v = \left\{\mathbf{x} \in \mathbb{R}^2\colon\ p(\mathbf{x}) \in \left[\frac{d}{2^{j+v}}, \frac{d}{2^{j+v-1}}\right]\right\}.$$

Let $\mathbf{a}_{1t}$ and $\mathbf{a}_{2t}$ be the points on the lines $y = 2x$ and $y = x/2$, respectively, such that

$$p(\mathbf{a}_{1t}) = p(\mathbf{a}_{2t}) = \frac{t}{3} \cdot \frac{d}{2^{j+v}}, \quad \text{with } 4 \leqslant t \leqslant 5.$$

Moreover, let $\mathbf{b}_{1t}$, $\mathbf{b}_{2t}$ and $\mathbf{c}_{1t}$, $\mathbf{c}_{2t}$ be the orthogonal projections of $\mathbf{a}_{1t}$ and $\mathbf{a}_{2t}$ on the horizontal line and on the vertical line through $\mathbf{o}$, respectively.

Finally let $\mathbf{d}_t = (\frac{t}{3} \cdot \frac{d}{2^{j+v}}, 0)$ and $\mathbf{f}_t = (0, \frac{t}{3} \cdot \frac{d}{2^{j+v}})$. Therefore, since

$$|\mathbf{o} - \mathbf{c}_{1t}| = |\mathbf{a}_{1t} - \mathbf{b}_{1t}| = 2 \cdot |\mathbf{o} - \mathbf{b}_{1t}| = 2 \cdot |\mathbf{a}_{1t} - \mathbf{c}_{1t}|,$$

$$|\mathbf{b}_{1t} - \mathbf{d}_t| = |\mathbf{a}_{1t} - \mathbf{b}_{1t}|, \quad \text{and} \quad |\mathbf{a}_{1t} - \mathbf{c}_{1t}| = |\mathbf{f}_t - \mathbf{c}_{1t}|,$$

then we have

$$\frac{t}{3} \cdot \frac{d}{2^{j+v}} = |\mathbf{o} - \mathbf{d}_t| = 3 \cdot |\mathbf{o} - \mathbf{b}_{1t}| \tag{7}$$

and

$$\frac{t}{3} \cdot \frac{d}{2^{j+v}} = |\mathbf{o} - \mathbf{f}_t| = 3 \cdot |\mathbf{a}_{1t} - \mathbf{c}_{1t}|. \tag{8}$$

Moreover, since

$$|\mathbf{a}_{2t} - \mathbf{c}_{2t}| = |\mathbf{o} - \mathbf{b}_{2t}| = 2 \cdot |\mathbf{a}_{2t} - \mathbf{b}_{2t}| = 2 \cdot |\mathbf{o} - \mathbf{c}_{2t}|,$$

$$|\mathbf{a}_{2t} - \mathbf{b}_{2t}| = |\mathbf{b}_{2t} - \mathbf{d}_t|, \quad \text{and} \quad |\mathbf{c}_{2t} - \mathbf{f}_t| = |\mathbf{a}_{2t} - \mathbf{c}_{2t}|,$$

then we have

$$\frac{t}{3} \cdot \frac{d}{2^{j+v}} = |\mathbf{o} - \mathbf{d}_t| = 3 \cdot |\mathbf{a}_{2t} - \mathbf{b}_{2t}|, \tag{9}$$

and also

$$\frac{t}{3} \cdot \frac{d}{2^{j+v}} = |\mathbf{o} - \mathbf{f}_t| = 3 \cdot |\mathbf{o} - \mathbf{c}_{2t}|. \tag{10}$$

Therefore, if $v \leqslant i - 1$, by (7) we have

$$|\mathbf{a}_{1t} - \mathbf{b}_{1t}| = 2 \cdot |\mathbf{o} - \mathbf{b}_{1t}| = \frac{t}{3^2} \cdot \frac{d}{2^{j+v-1}} > \frac{4}{3^2} \cdot \frac{d}{2^{j+v-1}} > \frac{d}{2^{j+v+1}} \geqslant \frac{d}{2^{j+i}} > a_2,$$

and by (10),

$$|\mathbf{a}_{2t} - \mathbf{c}_{2t}| = 2 \cdot |\mathbf{o} - \mathbf{c}_{2t}| = \frac{t}{3^2} \cdot \frac{d}{2^{j+v-1}} > \frac{4}{3^2} \cdot \frac{d}{2^{j+v-1}} \geqslant \frac{d}{2^{j+v+1}} \geqslant \frac{d}{2^{j+i}} > a_1.$$

Moreover, if $v \leqslant i - 2$, by (9) we infer

$$|\mathbf{a}_{2t} - \mathbf{b}_{2t}| = \frac{|\mathbf{o} - \mathbf{d}_t|}{3} = \frac{t}{3^2} \cdot \frac{d}{2^{j+v}} > \frac{4}{3^2} \cdot \frac{d}{2^{j+v}} \geqslant \frac{d}{2^{j+v+2}} \geqslant \frac{d}{2^{j+i}} > a_2,$$

and by (8),

$$|\mathbf{a}_{1t} - \mathbf{c}_{1t}| = \frac{|\mathbf{o} - \mathbf{f}_t|}{3} = \frac{t}{3^2} \cdot \frac{d}{2^{j+v}} > \frac{4}{3^2} \cdot \frac{d}{2^{j+v}} \geqslant \frac{d}{2^{j+v+2}} \geqslant \frac{d}{2^{j+i}} > a_1.$$

Consequently, the points $\mathbf{a}_{1t}$ and $\mathbf{a}_{2t}$ belong to the interval I^* as soon as we show that

$$|\mathbf{a}_{1t} - \mathbf{b}_{1t}| < b_2 \quad \text{and} \quad |\mathbf{a}_{2t} - \mathbf{b}_{2t}| < b_2;$$

$$|\mathbf{a}_{1t} - \mathbf{c}_{1t}| < b_1 \quad \text{and} \quad |\mathbf{a}_{2t} - \mathbf{c}_{2t}| < b_1.$$

First of all let as notice that

$$b_1 > \frac{d}{2^{j+3}} \quad \text{and} \quad b_2 > \frac{d}{2^{j+3}}.$$

Indeed the condition $b_1 \leqslant d/2^{j+3}$ easily implies $b_2 > d/2^{j+1}$. Then

$$b_2 - a_2 > \frac{d}{2^{j+1}} - \frac{d}{2^{j+i}} > \frac{d}{2^{j+3}} \geqslant b_1 > b_1 - a_1,$$

and, by the α-regularity of I^*, we get the following contradiction:

$$\frac{d}{2^{j+2}} = d \cdot \frac{2^{i-2}}{2^{j+i}} < d \cdot \left(\frac{1}{2^{j+1}} - \frac{1}{2^{j+i}}\right) < b_2 - a_2 \leqslant \frac{b_1 - a_1}{\alpha}$$
$$\leqslant 2(b_1 - a_1) < 2b_1 \leqslant \frac{d}{2^{j+2}}.$$

Analogously we can prove that the condition $b_2 \leqslant d/2^{j+3}$ is impossible. Therefore, if $v \geqslant 4$ we have

$$
\begin{aligned}
|\mathbf{a}_{1t} - \mathbf{b}_{1t}| &< \frac{5}{3^2} \cdot \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+3}} < b_2, \\
|\mathbf{a}_{2t} - \mathbf{b}_{2t}| &< \frac{5}{3^2} \cdot \frac{d}{2^{j+v}} < \frac{d}{2^{j+v}} < \frac{d}{2^{j+3}} < b_2, \\
|\mathbf{a}_{1t} - \mathbf{c}_{1t}| &< \frac{5}{3^2} \cdot \frac{d}{2^{j+v}} < \frac{d}{2^{j+v}} < \frac{d}{2^{j+3}} < b_1, \\
|\mathbf{a}_{2t} - \mathbf{c}_{2t}| &< \frac{5}{3^2} \cdot \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+3}} < b_1.
\end{aligned}
$$

In conclusion we have proved that, if $4 \leqslant v \leqslant i-2$, then the interval I^* contains the points $\mathbf{a}_{1t}, \mathbf{a}_{2t}$ for all $4 \leqslant t \leqslant 5$. This implies that I^* contains the intervals $[\mathbf{a}_{14}, \mathbf{a}_{15}]$ and $[\mathbf{a}_{24}, \mathbf{a}_{25}]$. Therefore I^* covers at least $(i-5)m$ of the r_{m+1} squares $Q_{(m+1)h}$, $1 \leqslant h \leqslant r_{m+1}$. Thus

$$
\mu_2\big([\mathbf{a}^*, \mathbf{b}^*]\big) \geqslant (i-5) \cdot m \cdot \frac{4}{(m+1)r_{m+1}} = (i-5) \cdot \frac{4m}{(m+1)^2} \cdot \omega_m^2;
$$

and, since $\beta \leqslant m$ and $\beta \leqslant i+2$, we have

$$
\begin{aligned}
\big|f(\mathbf{b}^*) - f(\mathbf{a}^*)\big|^2 &= \frac{\beta^2}{m^2} \cdot \omega \mathbf{a}_m^2 \leqslant \frac{\beta}{m} \cdot \frac{\beta}{m} \cdot \frac{(m+1)^2}{4m(i-5)} \mu_2\big([\mathbf{a}^*, \mathbf{b}^*]\big) \\
&< \frac{(i+2)(m+1)^2}{4(i-5)m^2} \mu_2\big([\mathbf{a}^*, \mathbf{b}^*]\big).
\end{aligned}
$$

Moreover, since $m \geqslant 9$, we have $5m^2 - 22m + 7 > 0$ and, since $i \geqslant 9$, we have also

$$
\begin{aligned}
i\big(3m^2 - 2m + 1\big) &\geqslant 27m^2 - 18m + 9 = 22m^2 + 4m + 2 + 5m^2 - 22m + 7 \\
&> 22m^2 + 4m + 2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(i+2)(m+1)^2 &= im^2 + 2im + i + 2m^2 + 4m + 2 \\
&= \big(4im^2 - 20m^2\big) - 3im^2 + 2im + i + 22m^2 + 4m + 2 \\
&= 4(i-5)m^2 - i\big(3m^2 - 2m - 1\big) + 22m^2 + 4m + 2 \\
&< 2(i-5)m^2.
\end{aligned}
$$

Consequently $|f(\mathbf{b}^*) - f(\mathbf{a}^*)|^2 < \mu_2([\mathbf{a}^*, \mathbf{b}^*])$. This completes the proof.

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